CHAPTER 17

MULTIPLE INTEGRATION

17.1 The Double Integral and Iterated Integral

PREREQUISITES

1. Recall the definition of the integral in one variable (Section 4.3).
2. Recall the basic rules of integration (Chapter 7).

PREREQUISITE QUIZ

1. Define an upper sum.
2. What is the relationship between lower sums, upper sums, and the integral?
3. Compute the following integrals:
   (a) $\int(x^3 + 2 + 1/x + \cos x)dx$
   (b) $\int x \sin x \, dx$
   (c) $\int(e^t + \sqrt{2t + 2})dt$
   (d) $\int_1^2 x^2 dx$

GOALS

1. Be able to define the double integral in terms of lower sums and upper sums.
2. Be able to evaluate double integrals over rectangular regions by using iterated integrals.

STUDY HINTS

1. Notation. $[a,b] \times [c,d]$ describes the rectangle $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$. As with the single variable notation, the use of parentheses rather than brackets means that the boundary is not included in the rectangle. A double integral over a rectangle is written as $\int_c^d\int_a^b f(x,y)dx\,dy$. The inner differential dx is associated with the inner limits a and b. Similarly, the outer differential is associated with the outer limits. Sometimes, it is written $\iint_D f(x,y)dx\,dy$ to indicate integration over the region $D = [a,b] \times [c,d]$.

2. Dummy variables. Just as the variable x was commonly used in one variable calculus, x and y are commonly used in two variable calculus. However, any other letter may also be used. For example, $\int_0^1\int_2^3 f(x,y)dx\,dy = \int_0^1\int_2^3 g(u,v)\,du\,dv$ if $f(x,y) = g(u,v)$.

3. Geometry. In one-variable integration, the integral represented the area under the curve, provided the integrand is nonnegative. With two variables, another dimension is added. The geometric interpretation is the volume under the surface, provided the integrand is nonnegative.

4. Definition. Again, the double integral is that unique number which falls between all upper and all lower sums.

5. Properties. The properties of double integrals are analogous to those of ordinary integrals. If you understand each statement, there should be no need to memorize them.

6. Iterated integrals. The iterated integral is simply the double integral written with brackets: $\int_c^d[\int_a^b f(x,y)dx]dy$ or $\int_a^b[\int_c^d f(x,y)dy]dx$. The brackets indicate that you evaluate the inner integral first and then the outer one. Notice that the order of integration doesn't matter.

7. Computing double integrals. This is just a reverse of partial differentiation. All variables are held constant except for the one upon which the integration is performed.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The integral over each subrectangle is the value of g times the area: $\iint g(x,y)dx\,dy = -2 \times 3 + 0 \times 1 + 1 \times 9 + 3 \times 3 = 12$.

5. $\int_0^3\int_0^2 x^3y\,dx\,dy = \int_0^3(x^4y/4)\big|_{x=0}^2 dy = \int_0^3 4y\,dy = 2y^2\big|_0^3 = 18$.

9. $\int_{-1}^1\int_0^1 ye^x dy\,dx = \int_{-1}^1(y^2e^x/2)\big|_{y=0}^1 dx = (1/2)\int_{-1}^1 e^x dx = (1/2)e^x\big|_{-1}^1 = e/2 - 1/2e$.

13. The integral is $\int_{-2}^2\int_0^1(x^2 + 2xy - y\sqrt{x})dx\,dy = \int_{-2}^2(x^3/3 + x^2y - (2/3)yx^{3/2})\big|_{x=0}^1 dy = \int_{-2}^2(1/3 - y/3)dy = (y/3 - y^2/6)\big|_{-2}^2 = 4/3$.

17. $\int_2^4\int_{-1}^1 x(1 + y)dx\,dy = \int_2^4(1 + y)dy\cdot\int_{-1}^1 x\,dx = (y + y^2/2)\big|_2^4\cdot(x^2/2)\big|_{-1}^1 = (8)(0) = 0$. This is what we expected from part (b) of Exercise 3.

21. The mass is the double integral over the density, as in Example 8. Let the center of the plate be located at the origin. Then $r^2 = x^2 + y^2$ and the mass is $\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(4 + x^2 + y^2)dx\,dy = \int_{-1/2}^{1/2}(4x + x^3/3 + xy^2)\big|_{x=-1/2}^{1/2} dy = \int_{-1/2}^{1/2}(49/12 + y^2)dy = (49y/12 + y^3/3)\big|_{-1/2}^{1/2} = 25/6$ grams.

25. (a) At time t on day T , $[1 - \sin^2\alpha\cos^2(2\pi T/365)]^{1/2}\cos(2\pi t/24)$ and $\sin\alpha\cos(2\pi T/365)$ are both constants. Denote these two constants by A and B , respectively. Thus, $I = A\cos\ell + B\sin\ell$; ℓ is in degrees and we would like to integrate in the same units. Colorado is 6° longitude wide, which is equivalent to 660 km or 8° latitude. The total solar energy is the double integral $\iint_D I\,dx\,d\ell$, where x is the east-west distance. Thus, the integrated solar energy is $8\int_{33}^{41}(A\cos\ell + B\sin\ell)d\ell = 8(A\sin\ell - B\cos\ell)\big|_{33}^{41} = 0.88A + 0.67B$.

25. (b) Integrating the result of part (a) from t_1 to t_2 is the total solar energy received in the state between the times t_1 and t_2 on day T .

SECTION QUIZ

1. (a) Compute $\int_0^2\int_1^4(x^3y + xy^2)dy\,dx$.

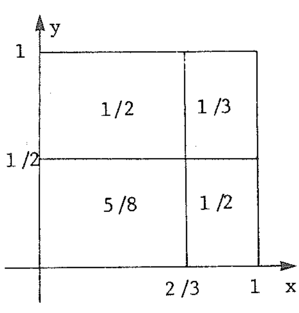
   (b) Explain the geometric interpretation of this integral.

2. The integral $\int_a^b\int_c^d xy\,dy\,dx$ is $(\int_a^b x\,dx)(\int_c^d y\,dy) = (x^2/2)|_a^b\cdot(y^2/2)|_c^d$.

   (a) Under what conditions can the above method of multiplying one-variable integrals be used to evaluate double integrals?

   (b) Use the technique described in part (a) to integrate $\int_a^b\int_c^d(x - 1)(y^2 + y)dy\,dx$ or explain why the method is inappropriate.

3. Find a formula for the volume between two surfaces $z = f(x,y)$ and $z = g(x,y)$ over the rectangle $[a,b] \times [c,d]$, assuming $f(x,y) > g(x,y)$.

4. 

A monster has begun to devour downtown Cleveland. The monster began its rampage by quickly eating up the block $0 \leq x \leq 1$, $0 \leq y \leq 1$. The four buildings on the block have height, width, and length as diagrammed. For example, one building has width $1/2$, length $2/3$, and height $5/8$.

   (a) If $f(x,y)$ is the height of the buildings, write the volume of the buildings on the block as a double integral.

   (b) How many cubic blocks did the monster consume when it devoured the first block of buildings?

ANSWERS TO PREREQUISITE QUIZ

1. In one-variable calculus, an upper sum is any number which is greater than the area under the graph of f.

2. The integral is precisely that number which lies between all upper sums and all lower sums.

3. (a) $x^4/4 + 2x + \ln|x| + \sin x + C$

   (b) $-x\cos x + \sin x + C$

   (c) $e^t + (2t + 2)^{3/2}/3 + C$

   (d) $7/3$

ANSWERS TO SECTION QUIZ

1. (a) 72

   (b) The volume between $f(x,y) = x^3y + xy^2$ and the xy-plane over the rectangle $[0,2] \times [1,4]$ is 72.

2. (a) When the integrand can be factored in functions of y only and x only, and the limits of integration are constant.

   (b) $\int_a^b(x - 1)dx \cdot \int_c^d(y^2 + y)dy = [(b^2 - a^2)/2 - (b - a)][(d^3 - c^3)/3 - (d^2 - c^2)/2]$

3. $\int_a^b\int_c^d[f(x,y) - g(x,y)]\,dy\,dx$

4. (a) $\int_0^1\int_0^1 f(x,y)\,dx\,dy$

   (b) $37/72$

17.2 The Double Integral Over General Regions

PREREQUISITES

1. Recall how to integrate by using iterated integrals (Section 17.1).

PREREQUISITE QUIZ

1. Compute the following integrals:

   (a) $\int_0^3\int_{-2}^0 xy^2\ dx\ dy$

   (b) $\int_0^3\int_{-2}^0 xy^2\ dy\ dx$

   (c) $\int_0^{\pi}\int_0^1 u\cos v\ du\ dv$

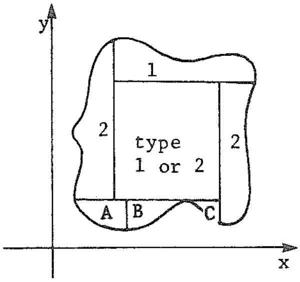
GOALS

1. Be able to integrate a double integral over a general planar region.
2. Be able to change the order of integration in double integrals and identify the corresponding region.

STUDY HINTS

1. Regions for theory. The notations D^*, D^{**}, f^*, f^{**} are introduced so that the transition can be made from rectangular regions to general regions. You won't need to worry about these concepts for solving most of the problems.
2. Region types. Type 1 regions are enclosed by two curves which are graphs of functions of x, and the lines $x = a$ and $x = b$. Type 2 regions are bounded by two curves which are graphs of functions of y, and the lines $y = a$ and $y = b$. See Figs. 17.2.3 and 17.2.4. Recognizing the region type will help you set up an integral. Naming the regions type 1 and type 2 is most important for discussion purposes. Your primary concern should be learning to perform multiple integration, not learning how to name the regions.

3. Simplifying complicated regions. Any planar region may be broken up into regions of type 1 and type 2. For example, the region at the left is divided into seven pieces, each of which is either type 1 or type 2. Fig. 17.2.5 and regions A, B, and C in the figure at the left show how some regions may be both type 1 and type 2.

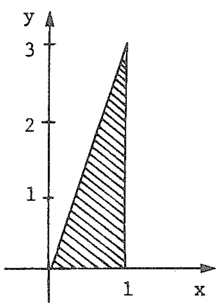


4. Integration method. For type 1 regions, it is best to integrate in y and then in x. For type 2 regions, integrate first in x, then y.

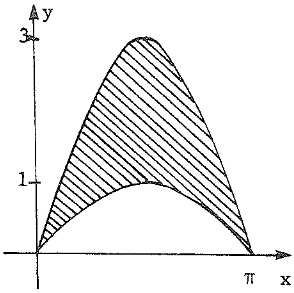
5. Choosing integration limits. When you perform multiple integration, be sure that the limits of integration do not include any previously integrated variables. In particular, no variable should appear in the limits of the outer most integral. For example, the integral of $(x + y)^2$ over the region D: $0 \leqslant y \leqslant x^2, 0 \leqslant x \leqslant 1$, should be written as $\int_0^1\int_0^{x^2} (x + y)^2 dy\,dx$, not $\int_0^{x^2}\int_0^1 (x + y)^2 dx\,dy$. Sketching the region often helps in choosing your limits.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

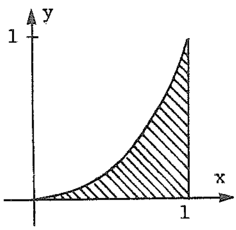
1. This is both a type 1 and a type 2 region. It can be described by x in $[0,1]$ and $\phi_1(x) = 0 \leqslant y \leqslant 3x = \phi_2(x)$. On the other hand, it can be described by y in $[0,3]$ and $\psi_1(y) = y/3 \leqslant x \leqslant 1 = \psi_2(y)$.



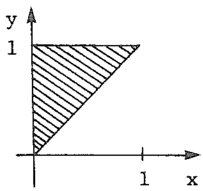
5. The region D is described by $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant x$. Thus, $\iint_D (x + y)^2 dx\,dy = \int_0^1\int_0^x (x + y)^2 dy\,dx = \int_0^1 [(x + y)^3/3]\big|_{y=0}^{x} dx = \int_0^1 (8x^3/3 - x^3/3)dx = \int_0^1 (7x^3/3)dx = (7x^4/12)\big|_0^1 = 7/12$.

9. 

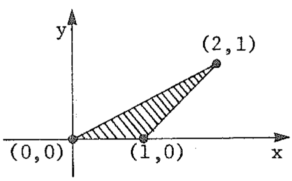
This region is type 1. $\int_0^\pi\int_{\sin x}^{3\sin x} x(1+y)\,dy\,dx = \int_0^\pi [x(1+y)^2/2]\Big|_{y=\sin x}^{3\sin x} dx = 2\int_0^\pi x(\sin x + 2\sin^2 x)dx$. Then, integration by parts yields $2[-x\cos x + \sin x + x^2/2 - x\sin 2x/2 - \cos 2x/2]\Big|_0^\pi = 2\pi + \pi^2$.

13. 

This is a type 1 region. The integral is $\int_0^1\int_0^{x^2}(x^2 + xy - y^2)dy\,dx = \int_0^1[(x^2y + xy^2/2 - y^3/3)\Big|_{y=0}^{x^2}]dx = \int_0^1(x^4 + x^5/2 - x^6/3)dx = (x^5/5 + x^6/12 - x^7/21)\Big|_0^1 = 1/5 + 1/12 - 1/21 = 33/140$.

17. 

When the order of integration is interchanged, we have $0 \leqslant y \leqslant 1$ and $0 \leqslant x \leqslant y$. Thus, the integral is $\int_0^1\int_0^y xy\,dx\,dy = \int_0^1 y(x^2/2)\Big|_{x=0}^{y} dy = \int_0^1(y^3/2)dy = (y^4/8)\Big|_0^1 = 1/8$.

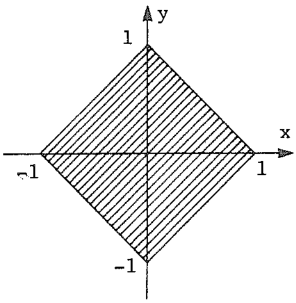
21. 

D is described by $0 \leqslant y \leqslant 1$ and $2y \leqslant x \leqslant y + 1$, so the integral is $\int_0^1\int_{2y}^{y+1}(x - y)\,dx\,dy = \int_0^1[(x^2/2 - xy)\Big|_{x=2y}^{y+1}]dy = \int_0^1[(1 - y^2)/2]dy = (1/2)(y - y^3/3)\Big|_0^1 = (1/2)(2/3) = 1/3$.

25. For a type 1 region, we have $a \leqslant x \leqslant b$ and $\phi_1(x) \leqslant y \leqslant \phi_2(x)$. Thus, $\iint_D dx\,dy = \int_a^b\int_{\phi_1(x)}^{\phi_2(x)} dy\,dx = \int_a^b[\phi_2(x) - \phi_1(x)]dx$, which is the formula for the area between the curves $\phi_2(x)$ and $\phi_1(x)$.

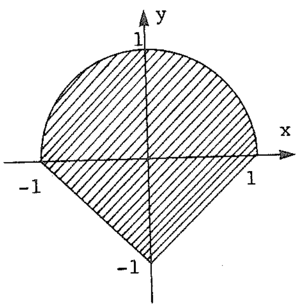
SECTION QUIZ

1.



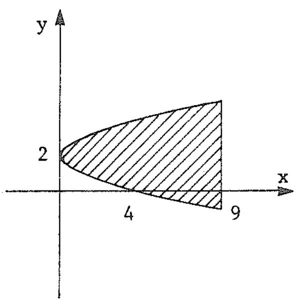
Integrate $x^2 + y^2$ over the square region shown in the figure.

2.



Integrate xy over the region shown in the figure.

3.



Integrate $2x + y$ over the parabolic region shown in the figure.

4. A region in the first quadrant is described by $0 \leqslant y \leqslant 2 - 2x$ and $0 \leqslant x \leqslant 1$. We wish to integrate $f(x,y) = x$ over this region.

(a) Sketch the region.

(b) Explain what is wrong with $\int_0^{2-2x}\int_0^1 x\,dx\,dy$.

(c) Compute the double integral.

5. The owner of a new amusement park wants his customers to begin enjoying themselves as soon as they leave their hotel rooms. Thus, he has built a water slide along the back wall of the hotel. The base of the hill on which the water slide is built is the semi-ellipse $x^2 + 4y^2/\pi^2 = 1$, $x \leqslant 0$. The height of the hill is $f(x,y) = x + \cos y$. The owner wants to know how much dirt he bought to build his water slide so he can deduct it from his income tax.

(a) Express the volume of the hill in the form $\int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$, but do not evaluate.

(b) Express the volume in the form $\int_c^d \int_{v(y)}^{u(y)} f(x,y)\,dx\,dy$, but do not evaluate.

ANSWERS TO PREREQUISITE QUIZ

1. (a) -18

   (b) 12

   (c) 0

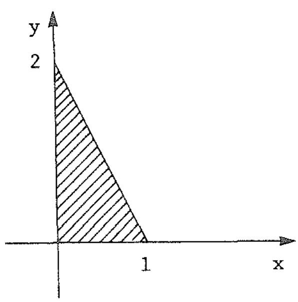
ANSWERS TO SECTION QUIZ

1. 2/3

2. 0

3. 460.8

4. (a)



4. (b) The outermost limits contain a variable which has already been integrated.

(c) 1/3

5. (a) $\int_{-1}^{0}\int_{-(\pi/2)\sqrt{1-x^2}}^{(\pi/2)\sqrt{1-x^2}} (x + \cos y)dy\, dx$

(b) $\int_{-\pi/2}^{\pi/2}\int_{-\sqrt{1-4y^2/\pi^2}}^{0} (x + \cos y)dx\, dy$

17.3 Applications of the Double Integral

PREREQUISITES

1. Recall how to compute a double integral (Sections 17.1 and 17.2).
2. Recall how to compute the average of a function of one variable (Section 9.3).
3. Recall how to find the center of mass for functions of one variable (Section 9.4).
4. Recall how to find surface areas of graphs of functions of one variable revolved about an axis (Section 10.3).

PREREQUISITE QUIZ

1. What is the average value of $f(x) = x^2$ over the interval $[0,1]$?
2. If $f(x) \geqslant 0$ on $[a,b]$, state the two integration formulas used to compute $\bar{x}$ and $\bar{y}$, the center of mass of the region under the graph of $f(x)$ on $[a,b]$.
3. If the graph of $f(x) \geqslant 0$ on $[a,b]$ is revolved around the x-axis, what formula is used to compute its surface area of revolution?
4. If the graph of $f(x) \geqslant 0$ on $[a,b]$ is revolved around the y-axis, what formula is used to compute its surface area of revolution?
5. Compute $\int_0^1\int_0^x 3\,dy\,dx$.

GOALS

1. Be able to compute the volume under a surface or the area of a region by using double integrals.
2. Be able to apply double integration to compute an average, a center of mass, or a surface area.

STUDY HINTS

1. Volume and area. Recall that the geometric interpretation of the double integral is the volume under the surface $z = f(x,y)$. If the height is $f(x,y) = 1$, then the integral represents the area of the region D: integral = volume = (area of base)(height) = area of D , since the height is 1 .

2. Example 1 factorization. You were asked to explain the factorization at the conclusion of Example 1. Here is one: $10^4/3^4\cdot2^4 + 10\cdot5^3/3^2\cdot2^3 - 5^4/3\cdot2^4 = 2^4\cdot5^4/3^4\cdot2^4 + 2\cdot5^4/3^2\cdot2^3 - 5^4/3\cdot2^4 = (2^4\cdot5^4 + 2^2\cdot3^2\cdot5^4 - 3^3\cdot5^4)/3^4\cdot2^4 = (16 + 36 - 27)5^4/3^4\cdot2^4 = 25(5^4)/3^4\cdot2^4 = 5^6/3^4\cdot2^4$. Sometimes manipulations like this can be easier than getting a big fraction as in the text.

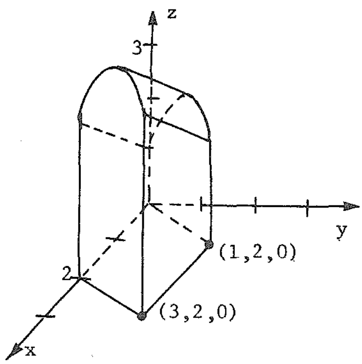
3. Average. In one-variable calculus, the average of a function was the integral divided by the length of the interval of integration. Now that we are dealing with an integral over a region, we divide by the area, so the average is the integral divided by the area of the region of integration.

4. Center of mass. Since mass is density times area, we have, for constant density, mass = $\rho\iint_D dx\,dy$. If density is a function of x and y , mass becomes $\iint_D\rho(x,y)dx\,dy$. By applying the infinitesimal argument of Section 9.4, we get $\bar{x} = \iint_D x\rho(x,y)dx\,dy/\iint_D \rho(x,y)dx\,dy$ and $\bar{y} = \iint_D y\rho(x,y)dx\,dy/\iint_D \rho(x,y)dx\,dy$.

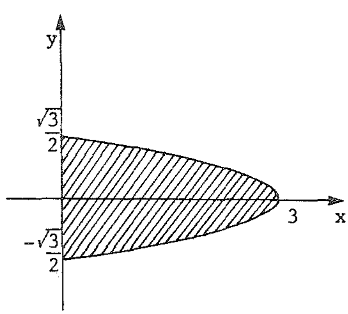
5. Surface area. The best thing to do is to memorize the formula in the box on p. 857. The integrand resembles the arc length integrand in one variable except that an extra differential term is under the radical.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

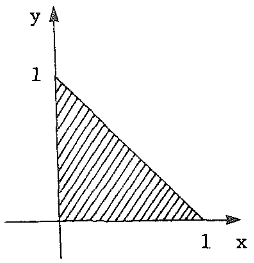
1. The volume is $\int_c^d\int_a^b f(x,y)dx\,dy$. In this case, it is $\int_\pi^{3\pi}\int_0^2(x \sin y + 3)dx\,dy = \int_\pi^{3\pi}[(x^2 \sin y/2 + 3x)|_{x=0}^{2}]dy = \int_\pi^{3\pi}(2 \sin y + 6)dy = (-2 \cos y + 6y)|_\pi^{3\pi} = 12\pi$.

5. 

The parallelogram is described by $0 \leqslant y \leqslant 1$ and $y/2 \leqslant x \leqslant y/2 + 2$, so the volume is $\int_0^2\int_{y/2}^{y/2+2}(1 + \sin(\pi y/2) + x)dx\,dy = \int_0^2[(x + x \sin(\pi y/2) + x^2/2)|_{x=y/2}^{y/2+2}]dy = \int_0^2(4 + 2\sin(\pi y/2) + y)dy = [4y - (4/\pi)\cos(\pi y/2) + y^2/2]|_0^2 = 10 + 8/\pi$.

9. 

The problem is to find the volume between the surface $z = x^3y$ and the xy-plane over the region shown in the figure. Due to the symmetry, we will only compute the volume above the xy-plane and then, multiply by 2 . The region on the xy-plane is described by $0 \leqslant y \leqslant \sqrt{3 - x}/2$ and $0 \leqslant x \leqslant 3$. Therefore, the half-volume is $\int_0^3\int_0^{\sqrt{3-x}/2} x^3y\,dy\,dx = \int_0^3(x^3y^2/2)\Big|_{y=0}^{\sqrt{3-x}/2} dx = (1/8)\int_0^3(3x^3 - x^4)dx = 243/160$. Thus, the entire volume is $243/80$.

13. 

The average value over a region is $\iint_D f(x,y)dx\,dy/\iint_D dx\,dy$. The region D is described by $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1 - x$, so the numerator is $\int_0^1\int_0^{1-x} e^{x+y}\,dy\,dx = \int_0^1(e^{x+y}|_{y=0}^{1-x})\,dx = \int_0^1(e - e^x)dx = (ex - e^x)|_0^1 = 1$. The denominator is $\int_0^1\int_0^{1-x} dy\,dx = \int_0^1(1 - x)dx = (x - x^2/2)|_0^1 = 1/2$, so the average value is $1/(1/2) = 2$.

17. The formula for the average value over a region is $\iint_D f(x,y)dx\,dy/\iint_D dx\,dy$. The numerator is $\int_0^a\int_0^a(x^2 + y^2)dx\,dy = \int_0^a[(x^3/3 + xy^2)|_{x=0}^a]\,dy = \int_0^a(a^3/3 + ay^2)dy = (a^3y/3 + ay^3/3)|_0^a = 2a^4/3$. The denominator is $\int_0^a\int_0^a dx\,dy = \int_0^a a\,dy = a^2$, so the average value is $(2a^4/3)/a^2 = 2a^2/3$.

21. The center of mass is given by $\bar{x} = \iint_D x\rho(x,y)dx\,dy/\iint_D\rho(x,y)dx\,dy$ and $\bar{y} = \iint_D y\rho(x,y)dx\,dy/\iint_D\rho(x,y)dx\,dy$. The disk is described by $-\sqrt{1 - (x - 1)^2} \leqslant y \leqslant \sqrt{1 - (x - 1)^2}$ and $0 \leqslant x \leqslant 2$. When $\rho(x,y) = x^2$, the following integrations will be used: $\int_{-1}^1 u^2\sqrt{1 - u^2}\,du = [(u/8)(2u^2 - 1)\sqrt{1 - u^2} + (1/8)\sin^{-1}u]|_{-1}^1 = \pi/8$; $\int_{-1}^1 u\sqrt{1 - u^2}\,du = (-1/3)(1 - u^2)^{3/2}|_{-1}^1 = 0$; $\int_{-1}^1\sqrt{1 - u^2}\,du = [(u/2)\sqrt{1 - u^2} + (1/2)\sin^{-1}u]|_{-1}^1 = \pi/2$. These formulas came from the integration table (numbers 38, 51, and 52). Also, let $v = 1 - u^2$, so $u^2 = 1 - v$, and $\int_{-1}^1 u^3\sqrt{1 - u^2}\,du = -\int_0^0(1 - v)\sqrt{v}(dv/2) = 0$. Therefore,

$$\bar{x} = \int_0^2\int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} x^2(x)\,dy\,dx\Big/\int_0^2\int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} x^2\,dy\,dx =$$

$$\int_0^2\left(yx^3\Big|_{y=-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}}\right)dx\Big/\int_0^2\left(yx^2\Big|_{y=-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}}\right)dx = 2\int_0^2 x^3\sqrt{1 - (x-1)^2}\,dx/$$

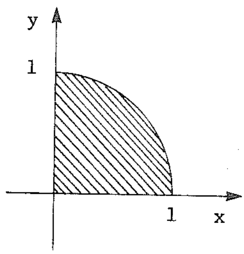
$2\int_0^2 x^2\sqrt{1 - (x - 1)^2}\,dx$. Let $u = x - 1$ to get $\int_{-1}^1(u + 1)^3\sqrt{1 - u^2}\,du/$

21. (continued)

$\int_{-1}^{1}(u+1)^2\sqrt{1-u^2}\,du = \int_{-1}^{1}(u^3+3u^2+3u+1)\sqrt{1-u^2}\,du/$
$\int_{-1}^{1}(u^2+2u+1)\sqrt{1-u^2}\,du = (0+3(\pi/8)+3(0)+\pi/2)/$
$(\pi/8+2(0)+\pi/2) = (7\pi/8)/(5\pi/8) = 7/5$. The numerator of $\bar{y}$ is

$$\int_0^2\int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} yx^2\,dy\,dx = \int_0^2\left(y^2x^2/2\Big|_{y=-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}}\right)dx = \int_0^2 0\,dx = 0 .$$

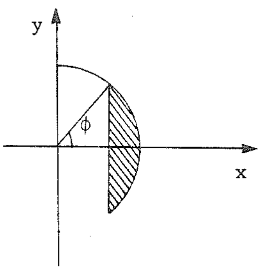
Thus, the center of mass is $(7/5,0)$.

25. 

We want the area of $z=\sqrt{x^2+y^2}$ lying above the region described by $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant \sqrt{1-x^2}$. $f_x = x/\sqrt{x^2+y^2}$ and $f_y = y/\sqrt{x^2+y^2}$. Thus, the surface area is

$$\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1+x^2/(x^2+y^2)+y^2/(x^2+y^2)}\,dy\,dx =$$

$\int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{2}\,dy\,dx = \int_0^1\sqrt{2}\sqrt{1-x^2}\,dx$. Using formula 38 of the integration table, this is $\sqrt{2}[(x/2)\sqrt{1-x^2}+(1/2)\sin^{-1}x]\Big|_0^1 = \sqrt{2}(\pi/4)$.

29. (a) 

Center the area around the x-axis. Then the spherical area lies above and below the region shown at the left. It can be described by $r\cos\phi \leqslant x \leqslant r$ and since $x^2+y^2=r^2$, we have $-\sqrt{r^2-x^2} \leqslant y \leqslant \sqrt{r^2-x^2}$. The equation of the sphere is $x^2+y^2+z^2=r^2$, so $z=\pm\sqrt{r^2-x^2-y^2}$. For the area lying above the xy-plane, $f_x = -x/\sqrt{r^2-x^2-y^2}$ and $f_y = -y/\sqrt{r^2-x^2-y^2}$. Thus, the upper area is

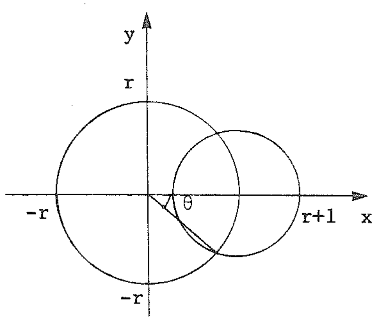
$$\int_{r\cos\phi}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\left(r/\sqrt{r^2-x^2-y^2}\right)dy\,dx .$$ Factor out $1/\sqrt{r^2-x^2}$

and let $u=y/\sqrt{r^2-x^2}$ to get $\int_{r\cos\phi}^{r}\int_{-1}^{1}(r/\sqrt{1-u^2}\,du\,dx =$

29. (a) (continued)

$\int_{r\cos\phi}^{r}(r\sin^{-1}u|_{-1}^{1})dx = \pi r\int_{r\cos\phi}^{r} dx = \pi r^2(1 - \cos\phi)$. Since there is also a bottom half, the entire area is $2\pi r^2(1 - \cos\phi)$.

(b)



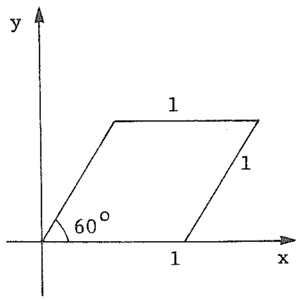
Center the larger circle at the origin, and center the smaller one at $(r,0)$, then the equations of the circles are $x^2 + y^2 = r^2$ and $(x - r)^2 + y^2 = 1$. Subtracting one equation from the other gives us the point of intersection: $x = (2r^2 - 1)/2r$. Therefore, $\cos\theta = (2r^2 - 1)/2r^2$, and by the formula derived in part (a), the surface area is $2\pi r^2(1 - (2r^2 - 1)/2r^2) = 2\pi r^2(1/2r^2) = \pi$. This is surprising because no matter how large r is, the surface area cut out is always π , independent of r .

33. As explained in the solution to Example 5, $z = \pm\sqrt{[f(x)]^2 - y^2}$ and the region of integration is given by $a \leqslant x \leqslant b$ and $-f(x) \leqslant y \leqslant f(x)$. The double integral of z over D gives the volume under the graph of z and since half of z lies below the xy-plane, the volume is $2\int_a^b\int_{-f(x)}^{f(x)} \sqrt{[f(x)]^2 - y^2}\, dy\, dx$. By formula 38 of the integration table, we get $2\int_a^b\{[(y/2)\sqrt{[f(x)]^2 - y^2} + ((f(x))^2/2)\sin^{-1}(y/f(x))]|_{y=-f(x)}^{f(x)}\}dx = 2\int_a^b((f(x))^2/2)(\pi/2 - (-\pi/2))\, dx = \pi\int_a^b[f(x)]^2dx$. This is exactly the same formula that was derived using one-variable calculus.

SECTION QUIZ

1. Find the volume of the solid bounded by $z = xy + 3$, the xy-plane, and the cylinder $(x - 2)^2 + (y - 2)^2 = 1$.

2. Find the average value of the unit hemisphere which lies above the xy-plane.

3. 

If the density is xy, find the center of mass of the parallelogram sketched at the left.

4. Find the surface area of $z = x + y + 4$ over the region $y \geqslant x^2$ and $y \leqslant 9$.

5. Herbie, the health nut, refuses to drink anything except melted snow water. Even when the weather is warm, Herbie imports snow from the North Pole. Last night, a blizzard left a fresh blanket of snow on the ground with height $2 + (x \sin y)/100$ in Herbie's backyard: $[0,100] \times [0,50]$.

   (a) If one cubic unit represents one gallon of water, how many gallon containers are needed to collect the water for Health Nut Herbie?

   (b) Herbie decides to cover the snow and collect it later. What size tarp does he need? Express your answer as a double integral.

ANSWERS TO PREREQUISITE QUIZ

1. 1/3

2. $\bar{x} = \int_a^b x\, f(x)dx / \int_a^b f(x)dx$; $\bar{y} = (1/2)\int_a^b [f(x)]^2 dx / \int_a^b f(x)dx$

3. $2\pi\int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx$

4. $2\pi\int_a^b x\sqrt{1 + [f'(x)]^2}\, dx$

5. 3/2

ANSWERS TO SECTION QUIZ

1. 11π

2. 2/3

3. $(19/20, 7\sqrt{3}/20)$

4. $12\sqrt{3}$

5. (a) $10050 - 50 \cos 50$

   (b) $(1/100)\int_0^{100}\int_0^{50} [10000 + \sin^2 y + x^2\cos^2 y]^{1/2} dy\, dx$

17.4 Triple Integrals

PREREQUISITES

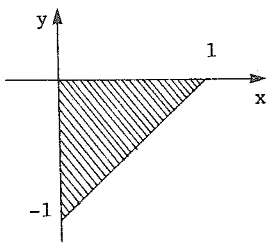
1. Recall how to compute double integrals (Section 17.2).

PREREQUISITE QUIZ

1. Compute the following integrals:

   (a) $\int_0^2 \int_0^{x^2} xy\, dy\, dx$

   (b) $\int_1^2 \int_{-y}^1 y\, dx\, dy$

2. 

   Compute $\iint_D (x + y)dy\, dx$ for the region D shown at the left.

GOALS

1. Be able to evaluate a triple integral by using iterated integrals.
2. Be able to change the order of integration in triple integrals.

STUDY HINTS

1. Triple integrals. As with all of the integration theory presented up to this point, the triple integral is that unique number that lies between all lower sums and all upper sums.
2. Iterated integral. As with double integrals, the triple integral may be reduced to iterated integrals. Again, the variables may be integrated in any order.

3. Region types. W is type I if it is bounded by two surfaces which are functions of x and y. The region D over which W falls in the xy-plane may be either type 1 or 2. Type II regions interchange the roles of x and z. Type III regions interchange the roles of y and z. As with double integrals, knowing the region type aids in choosing the limits of integration; however, knowing the name of the region type does not help in doing computations.

4. Balls. Example 3 demonstrates that the unit ball is a type I region. Written as $-1 \leqslant z \leqslant 1, -\sqrt{1 - z^2} \leqslant y \leqslant \sqrt{1 - z^2}$, and $-\sqrt{1 - z^2 - y^2} \leqslant x \leqslant \sqrt{1 - z^2 - y^2}$, the ball is a type II region. Similarly, the type III ball is described by $-1 \leqslant x \leqslant 1$, $-\sqrt{1 - x^2} \leqslant z \leqslant \sqrt{1 - x^2}$, $-\sqrt{1 - x^2 - z^2} \leqslant y \leqslant \sqrt{1 - x^2 - z^2}$. The solution of Example 3 uses a type 1 description for D. Using a type 2 description would generate three more descriptions of W, the unit ball.

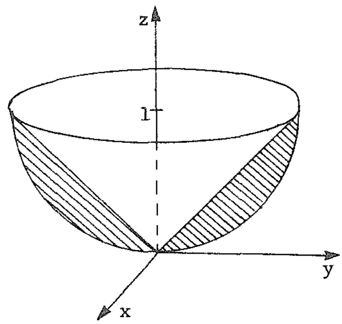
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The first method we use is to integrate with respect to x, then y, then z. The second method is integrating with respect to y, then z, then x. Any other order may be used and the answer is always the same.

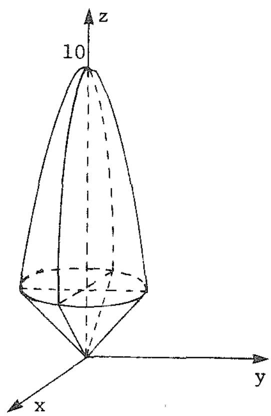
$$\iiint_W (2x + 3y + z)dx\,dy\,dz = \int_0^1\int_{-1}^1\int_1^2 (2x + 3y + z)dx\,dy\,dz =$$
$$\int_0^1\int_{-1}^1 (x^2 + 3xy + xz)\Big|_{x=1}^2 dy\,dz = \int_0^1\int_{-1}^1 (3 + 3y + z)dy\,dz =$$
$$\int_0^1 (3y + 3y^2/2 + yz)\Big|_{y=-1}^1 dz = \int_0^1 (6 + 2z)dz = (6z + z^2)\Big|_0^1 = 7 .$$

The second method gives $\int_1^2\int_0^1\int_{-1}^1 (2x + 3y + z)\,dy\,dz\,dx =$

$$\int_1^2\int_0^1 (2xy + 3y^2/2 + yz)\Big|_{y=-1}^1 dz\,dx = \int_1^2\int_0^1 (4x + 2z)dz\,dx =$$
$$\int_1^2 (4xz + z^2)\Big|_{z=0}^1 dx = \int_1^2 (4x + 1)dx = (2x^2 + x)\Big|_1^2 = 7 .$$

5. 

The region is sketched at the left. It lies over the region D, which is a circle of radius 1 centered at the origin. Thus, the region W can be described by $-1 \leqslant x \leqslant 1$, $-\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}$, and $x^2+y^2 \leqslant z \leqslant \sqrt{x^2+y^2}$. As described, W is type I.

9. 

The volume is $\iiint_W dx\,dy\,dz$. The projection of the intersection of the two surfaces is $x^2+y^2 = 10 - x^2 - 2y^2$, i.e., $x^2/(\sqrt{5})^2 + y^2/(\sqrt{10/3})^2 = 1$. Thus, the volume equals

$$\int_{-\sqrt{5}}^{\sqrt{5}}\int_{-\sqrt{(10-2x^2)/3}}^{\sqrt{(10-2x^2)/3}}\int_{x^2+y^2}^{10-x^2-2y^2} dz\,dy\,dx =$$

$$\int_{-\sqrt{5}}^{\sqrt{5}}\int_{-\sqrt{(10-2x^2)/3}}^{\sqrt{(10-2x^2)/3}}(10 - 2x^2 - 3y^2)dy\,dx =$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}}(10y - 2x^2y - y^3)\Big|_{y=-\sqrt{(10-2x^2)/3}}^{\sqrt{(10-2x^2)/3}} dx = 4\int_{-\sqrt{5}}^{\sqrt{5}}((10-2x^2)/3)^{3/2}dx =$$

$$4(2/3)^{3/2}\int_{-\sqrt{5}}^{\sqrt{5}}(5-x^2)^{3/2}dx = 4(2/3)^{3/2}[(x/4)(5-x^2)^{3/2} +$$

$$(15/4)((x/2)\sqrt{5-x^2} + (5/2)\sin^{-1}(x/\sqrt{5}))]\Big|_{-\sqrt{5}}^{\sqrt{5}} = 25\sqrt{2/3}\,\pi .$$

13. $\int_0^1\int_1^2\int_2^3 \cos[\pi(x + y + z)]\,dx\,dy\,dz = \int_0^1\int_1^2 [\sin(\pi(x + y + z))/\pi]\,\Big|_{x=2}^3\,dy\,dz =$
$(1/\pi)\int_0^1\int_1^2 [\sin(\pi(3 + y + z)) - \sin(\pi(2 + y + z))]\,dy\,dz =$
$(1/\pi)\int_0^1 [-\cos(\pi(3 + y + z))/\pi + \cos(\pi(2 + y + z))/\pi]\,\Big|_{y=1}^2\,dz =$
$(1/\pi^2)\int_0^1 [-\cos(\pi(5 + z)) + 2\cos(\pi(4 + z)) - \cos(\pi(3 + z))]\,dz =$
$(1/\pi^2)[-\sin(\pi(5 + z))/\pi + 2\sin(\pi(4 + z))/\pi - \sin(\pi(3 + z))/\pi]\,\Big|_0^1 = 0$.

17. $\int_0^\pi\int_0^1\int_0^{1-y} x^2 \cos z\,dx\,dy\,dz = \int_0^\pi\int_0^1 [(x^3 \cos z)/3\,\Big|_{x=0}^{1-y}]\,dy\,dz =$
$(1/3)\int_0^\pi\int_0^1 (1 - y)^3 \cos z\,dy\,dz$. Let $u = 1 - y$ to get
$(-1/3)\int_0^\pi\int_{-1}^0 u^3 \cos z\,du\,dz = (-1/3)\int_0^\pi [(u^4 \cos z)/4\,\Big|_{u=-1}^0]\,dz =$
$(1/12)\int_0^\pi \cos z\,dz = (1/12)(-\sin z)\,\Big|_0^\pi = 0$. (This problem could have been done in one step by integrating in z first.)

21. Let the box W be $[a,b] \times [c,d] \times [p,q]$, then
$\int_a^b\int_c^d\int_p^q F(x,y)\,dz\,dy\,dx$. When integrating in z , $F(x,y)$ is held constant, so the integral is $\int_a^b\int_c^d [zF(x,y)\,\Big|_{z=p}^q]\,dy\,dx =$
$(q - p)\int_a^b\int_c^d F(x,y)\,dy\,dz$. Thus, the triple integral of f over a box W is the double integral of F over the base of the box, times the height of the box.

25. Let W be the solid whose volume we wish to compute. Then the volume is $\iiint_W dx\,dy\,dz = \iiint_W dz\,dx\,dy$ by changing the order of integration. Using either formula (3) or (4), the integral becomes $\iint_D\int_0^{f(x,y)} dz\,dx\,dy =$
$\iint_D (z\,\Big|_{z=0}^{f(x,y)})\,dx\,dy = \iint_D f(x,y)\,dx\,dy$, which is the double integral of f over D .

SECTION QUIZ

1. Evaluate $\iiint_W (xz + yz)\,dx\,dy\,dz$, where W is the box $[0,1] \times [1,3] \times [-1,1]$.

2. (a) Evaluate $\int_1^2 \int_{-2}^1 \int_0^3 dy\, dz\, dx$.

   (b) Sketch the region of integration.

   (c) Give a geometric interpretation of the integral in part (a).

3. Define the triple integral in terms of upper and lower sums.

4. Express the volume between $z = xy$ and the region $[-3,2] \times [-1,1]$ on the xy-plane as a sum of triple integrals.

5. Suppose a college student's state of alertness during a lecture is a function of s , how much sleep he or she had the night before, in hours; i , the interest level of the professor's lecture; and g , the student's grade point average. A student's Falling Asleep Index for Lectures (FAIL) is determined by $1 - [\int_0^a \int_0^b \int_0^c s^2 ig\, dg\, di\, ds] / [\int_0^{10} \int_0^{10} \int_0^4 s^2 ig\, dg\, di\, ds]$.

   (a) What is a student's FAIL if $a = 5$, $b = 2$, and $c = 3.5$?

   (b) If $b = 2$ and $c = 3.5$ throughout the semester, how many hours of sleep is needed to keep FAIL above 0.5 ? (A FAIL of 0.5 indicates a 50% probability of falling asleep. A greater FAIL indicates a greater likelihood of falling asleep.)

ANSWERS TO PREREQUISITE QUIZ

1. (a) 16/3

   (b) 23/6

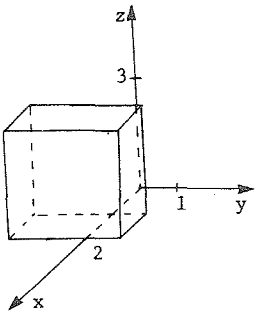
2. 0

ANSWERS TO SECTION QUIZ

1. 0

2. (a) 9

(b)



(c) It is the volume of the box sketched in part (b).

3. The precise number between all lower sums and all upper sums.

4. $\int_0^2\int_0^1\int_0^{xy} dz\ dy\ dx + \int_{-3}^0\int_{-1}^0\int_0^{xy} dz\ dy\ dx + \int_{-3}^0\int_0^1\int_{xy}^0 dz\ dy\ dx + \int_0^2\int_{-1}^0\int_{xy}^0 dz\ dy\ dx$

5. (a) 12751/12800

(b) $200\sqrt[3]{10/49}$ (Impossible! The student will probably fall asleep.)

17.5 Integrals in Polar, Cylindrical, and Spherical Coordinates

PREREQUISITES

1. Recall how to convert between polar and cartesian coordinates in the plane (Section 5.1).
2. Recall how to convert between cartesian, cylindrical, and spherical coordinates in space (Section 14.5).
3. Recall how to compute double and triple integrals (Sections 17.2 and 17.4).

PREREQUISITE QUIZ

1. Let $P = (2,1,-1)$ be the cartesian coordinates of a point in space.
   (a) What are the cylindrical coordinates of P ?
   (b) What are the spherical coordinates of P ?
2. Evaluate the triple integral $\int_0^1\int_0^{\pi/2}\int_0^{\pi/4} \rho^2 \sin\theta \cos(\phi/2)\, d\theta\, d\phi\, d\rho$.
3. Evaluate the double integral $\int_0^{\pi/2}\int_0^1 r^2 \cos\theta\, dr\, d\theta$.

GOALS

1. Be able to integrate multiple integrals in polar coordinates, cylindrical coordinates, and spherical coordinates.
2. Be able to evaluate the Gaussian integral.

STUDY HINTS

1. Polar coordinates. Recall that $dx\,dy$ converts to $r\,dr\,d\theta$ (if you remember that arc length is $r\,d\theta$ and draw Fig. 17.5.1, you can always remember this), so if we describe D in terms of r and θ to get D' and substitute, the resulting expression is $\iint_D f(x,y)dx\,dy = \iint_{D'} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$.

2. Gaussian integral. You should memorize the fact that $\int_{-\infty}^{\infty}\exp(-x^2)dx = \sqrt{\pi}$. In addition, you should be able to manipulate this equation to compute similar integrals. See Example 2.

3. Cylindrical coordinates. As with polar coordinates, $dx\,dy$ changes to $r\,dr\,d\theta$, so if we describe W in terms of r , θ , and z to get W' and substitute, the resulting expression is $\iiint_W f(x,y,z)dx\,dy\,dz = \iiint_{W'} f(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz$.

4. Spherical coordinates. You should memorize the fact that $dx\,dy\,dz$ converts to $\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ (the volume of the box in Fig. 17.5.4). By expressing W in terms of ρ , θ , and ϕ to get W' , substitution gives $\iiint_W f(x,y,z)dx\,dy\,dz = \iiint_{W'} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$. When you set up the limits of integration, remember that ϕ is measured from the "north pole," not from the "equator." Also, recall that θ is measured in a counterclockwise direction.

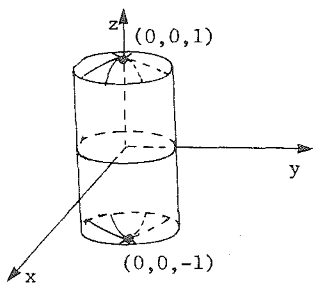
5. Choosing an integration method. If $x^2 + y^2$ occurs in the integrand of a double integral, it is generally a good idea to use polar coordinates. Cylindrical coordinates are generally used if $x^2 + y^2$ appears in the integrand of a triple integral. And finally, if $x^2 + y^2 + z^2$ appears in a triple integral, it is wise to try spherical coordinates.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The region D can be described by $0 \leqslant r \leqslant 2$, $0 \leqslant \theta \leqslant 2\pi$; thus $\iint_D (x^2 + y^2)^{3/2}\,dx\,dy = \int_0^{2\pi}\int_0^2 r(r^2)^{3/2}\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^4\,dr\,d\theta = \int_0^{2\pi}(r^5/5)\big|_{r=0}^{2\pi}\,d\theta = \int_0^{2\pi}(2^5/5)d\theta = 64\pi/5$.

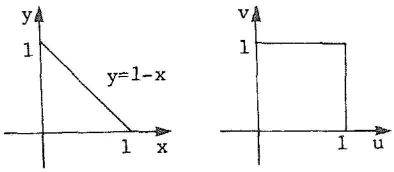
5. The disk is described by $0 \leqslant r \leqslant 4$ and $0 \leqslant \theta \leqslant 2\pi$, so $\iint_D(x^2 + y^2)dx\,dy = \int_0^{2\pi}\int_0^4(r^2)r\,dr\,d\theta = \int_0^{2\pi}(r^4/4)\big|_{r=0}^{4}\,d\theta = \cancel{(1/4)}\,4^3\int_0^{2\pi}d\theta = \cancel{\pi/2}$. 128π

9. can be described in spherical coordinates by $0 \leqslant \rho \leqslant 1$, $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \theta \leqslant 2\pi$; thus $\iiint_W (dx\,dy\,dz/\sqrt{1+x^2+y^2+z^2}) = \int_0^1\int_0^\pi\int_0^{2\pi} (\rho^2 \sin\phi \, d\theta\, d\phi\, d\rho/\sqrt{1+\rho^2}) = \int_0^1 [(2\pi\rho^2/\sqrt{1+\rho^2})\int_0^\pi \sin\phi\, d\phi]\,d\rho = \int_0^1 (4\pi\,\rho^2/\sqrt{1+\rho^2})d\rho = 4\pi\int_0^1 [(\rho^2+1-1)/\sqrt{1+\rho^2}]\,d\rho = 4\pi[\int_0^1 \sqrt{1+\rho^2}\,d\rho - \int_0^1 (d\rho/\sqrt{1+\rho^2})] = 4\pi[(\rho/2)\sqrt{1+\rho^2} + (1/2)\ln(\rho+\sqrt{1+\rho^2}) - \ln(\rho+\sqrt{1+\rho^2})]\big|_0^1 = 2\pi[\sqrt{2} - \ln(1+\sqrt{2})]$.

13. 

The region is described by $-\sqrt{1-x^2-y^2} \leqslant z \leqslant \sqrt{1-x^2-y^2}$, $-\sqrt{1/4-x^2} \leqslant y \leqslant \sqrt{1/4-x^2}$, and $-1/2 \leqslant x \leqslant 1/2$; thus, the volume is $\int_{-1/2}^{1/2}\int_{-\sqrt{1/4-x^2}}^{\sqrt{1/4-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = \int_{-1/2}^{1/2}\int_{-\sqrt{1/4-x^2}}^{\sqrt{1/4-x^2}} (2\sqrt{1-x^2-y^2})\,dy\,dx$. The region $-\sqrt{1/4-x^2} \leqslant y \leqslant \sqrt{1/4-x^2}$ and $-1/2 \leqslant x \leqslant 1/2$ can be described in polar coordinates by $0 \leqslant r \leqslant 1/2$ and $0 \leqslant \theta \leqslant 2\pi$. Therefore, we have volume $= \int_0^{2\pi}\int_0^{1/2} 2\sqrt{1-r^2}\, r\, dr\, d\theta = \int_0^{2\pi} [-(2/3)(1-r^2)^{3/2}]\big|_0^{1/2}\, d\theta = (\pi/6)(8-3\sqrt{3})$.

17. We know that $\int_{-\infty}^{\infty} c\exp(-x^2/\sigma)dx = c\int_{-\infty}^{\infty}\exp(-x^2/\sigma)dx$. Let $y = x/\sqrt{\sigma}$, so the integral becomes $\lim_{a\to\infty} c\int_{-a}^{a}\exp(-x^2/\sigma)dx = \lim_{a\to\infty} c\int_{-a/\sqrt{\sigma}}^{a/\sqrt{\sigma}}\exp(-y^2)dy\sqrt{\sigma} = \lim_{a\to\infty} c\sqrt{\sigma}\int_{-a/\sqrt{\sigma}}^{a/\sqrt{\sigma}}\exp(-y^2)dy$. By the Gaussian integral, this is $c\sqrt{\sigma}\int_{-\infty}^{\infty}\exp(-y^2)dy = c\sqrt{\pi\sigma}$. Therefore, the normalizing constant is $c = 1/\sqrt{\pi\sigma}$.

21. 

To graph the region in the uv-plane, note that when $x = 0$, $u = y$, so $u = uv$, which implies $u = 0$ or $v = 1$. So when $v = 1$, $u = y$ is in the interval $0 \leqslant u \leqslant 1$. When $y = 0$, $u = x$ and $0 = uv$, which implies $u = 0$ or $v = 0$ for $0 \leqslant u \leqslant 1$. When $y = 1 - x$, $u = x + (1 - x) = 1$ and $y = v$ for $0 \leqslant v \leqslant 1$. Thus, the region of integration is $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant 1$.

The integrand is $\exp[y/(x + y)] = \exp[uv/u] = e^v$. Rearrangement gives $x = u - uv$ and $y = uv$, so $\| \partial(x,y)/\partial(u,v) \| = \left| \begin{matrix} 1 - v & -u \\ v & u \end{matrix} \right| = u$. Therefore, the integral is $\int_0^1\int_0^1 e^v(u)\, du\, dv = \int_0^1 [(u^2/2)e^v|_{u=0}^1]\, dv = (1/2)e^v|_0^1 = (e - 1)/2$.

SECTION QUIZ

1. Discuss the evaluation of $\int_{-\infty}^{\infty} \exp(3x^2)\, dx$.

2. Evaluate the triple integral $\iiint_W ((x^2 + y^2)/z) dx\, dy\, dz$, where W is the cylinder $x^2 + y^2 \leqslant 9$ and $1 \leqslant z \leqslant 2$.

3. Evaluate the triple integral $\iiint_W ((x^2 + y^2)/(x^2 + y^2 + z^2)) dz\, dx\, dy$, where W is the sphere $x^2 + y^2 + z^2 \leqslant 4$.

4. Evaluate the double integral $\iint_D x\sqrt{x^2 + y^2}\, dy\, dx$, where D is the disk $x^2 + y^2 \leqslant 1$.

5. The big bad wolf lived in the center of the forest. Since the big bad wolf's big, sharp teeth were used to eat rabbits, the density of the rabbit population was $x^4 + 2x^2y^2 + y^4$ at (x,y). Compute the rabbit population within radius R of the big bad wolf's home, located at $(0,0)$.

ANSWERS TO PREREQUISITE QUIZ

1. (a) $(\sqrt{5}, \tan^{-1}(1/2), -1) \approx (2.24, 0.46, -1)$

   (b) $(\sqrt{6}, \tan^{-1}(1/2), \cos^{-1}(-1/\sqrt{6})) \approx (2.45, 0.46, 1.99)$

2. $-1/3$

3. $1/3$

ANSWERS TO SECTION QUIZ

1. The integral is infinite. The Gaussian integral requires a negative exponent.

2. $(81\pi/2) \ln 2$

3. $64\pi/3$

4. 0

5. $\pi R^6/6$

17.6 Applications of Triple Integrals

PREREQUISITES

1. Recall how to compute an average or center of mass using double integration (Section 17.3).
2. Recall how to compute triple integrals in cartesian coordinates (Section 17.4).
3. Recall how to compute triple integrals in spherical coordinates (Section 17.5).

PREREQUISITE QUIZ

1. Find the average of $f(x,y) = xy$ over the rectangle $[0,1] \times [0,2]$.
2. Compute the center of mass for the plate with density $\rho(x,y) = xy$ if the plate can be described by $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 2$.
3. Evaluate $\int_0^1\int_0^1\int_{-1}^1 z(x^2 + y^2)dx\,dy\,dz$.
4. Evaluate $\iiint_W z\,dx\,dy\,dz$, where W is the unit sphere.

GOALS

1. Be able to apply triple integrals for computing volumes, centers of mass, and averages.

STUDY HINTS

1. Volume. If W is the region under the graph of $z = f(x,y)$, then $\iiint_W dx\,dy\,dz = \iint_D z\,dx\,dy$. Since $z = f(x,y)$, the original integral becomes $\iint_D f(x,y)dx\,dy$, which is the formula given in the first half of this chapter.

2. Mass. $\iiint_W \rho(x,y,z)\,dx\,dy\,dz$ is just an extension of the one- and two-variable integrals.

3. Center of mass. The formulas are just an extension of the one- and two-variable cases.

4. Averages. With the addition of an extra variable, we now divide the integral by its volume, rather than the area for the two-variable case.

5. Gravitational potential. The chapter ends with a discussion of the application of triple integrals to gravity. Although you don't need to understand the physical theory behind the discussion, the mathematics should be clear to you. You probably will not need to reproduce the discussion for an exam.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The mass is given by $\iiint_W \rho(x,y,z)dx\,dy\,dz$.

   (a) Let ρ be the constant mass density. Then, the mass is $\rho\int_0^{1/2}\int_0^1\int_0^2 dz\,dy\,dx = \rho(1/2)(1)(2) = \rho$.

   (b) The mass is $\int_0^{1/2}\int_0^1\int_0^2 (x^2 + 3y^2 + z^2 + 1)dz\,dy\,dx =$ $\int_0^{1/2}\int_0^1 (zx^2 + 3y^2z + z^3/3 + z)\big|_{z=0}^{2} dy\,dx = \int_0^{1/2}\int_0^1 (2x^2 + 6y^2 + 14/3)dy\,dx = \int_0^{1/2} (2yx^2 + 2y^3 + 14y/3)\big|_{y=0}^{1} dx = \int_0^{1/2} (2x^2 + 20/3)dx =$ $(2x^3/3 + 20x/3)\big|_0^{1/2} = 1/12 + 10/3 = 41/3$.

5. Let $I = \iiint_W z\,dx\,dy\,dz$. Then, by considering the hemisphere as a type I region, we integrate with respect to x last, so $I =$ $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx = (1/2)\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)dy\,dx =$ $(1/2)\int_{-1}^{1} (4/3)(1 - x^2)^{3/2}\,dx$ (formula 38 of the integration table) $=$ $(2/3)[(x/4)(1 - x^2)^{3/2} + (3/4)(x\sqrt{1 - x^2} + (1/2)\sin^{-1}x]\big|_{-1}^{1}$ (formula 39) $= \pi/4$.

9. The average value of $f(x,y,z)$ is $\iiint_W f(x,y,z)dx\,dy\,dz/\iiint_W dx\,dy\,dz$. The numerator is $\int_0^2\int_0^4\int_0^6 \sin^2\pi z \cos^2\pi x\,dz\,dy\,dx = \int_0^2\int_0^4\int_0^6((1-\cos 2\pi z)/2) \times ((1+\cos 2\pi x)/2)dz\,dy\,dx = (1/4)\int_0^2(1+\cos 2\pi x)dx\cdot\int_0^4 dy\cdot\int_0^6(1-\cos 2\pi z)dz = (1/4)(x+\sin 2\pi x/2\pi)\big|_0^4\cdot 4\cdot(z-\sin 2\pi z/2\pi)\big|_0^6 = (1/4)(4)(4)(6) = 24$. The denominator is $\int_0^2\int_0^4\int_0^6 dz\,dy\,dx = (2)(4)(6) = 48$. Thus, the average value is $1/2$.

13. Using the method of Example 6, we have $V = GM/R = (6.67 \times 10^{-11}) \times (3 \times 10^{26})/(2 \times 10^8) = 1.00 \times 10^8 (m/s)^2$.

17. We will place (x_1,y_1,z_1) at $(0,0,R)$ and the spherically symmetric body will be centered at the origin. According to the text, the gravitational potential due to the region between the concentric spheres $\rho = \rho_1$ and $\rho = \rho_2$ is GM/R if (x_1,y_1,z_1) lies outside the attracting body. In this problem, the density is a function of the radius, so we will integrate over the regions between the spheres $\rho = \rho_0$ and $\rho = \rho_0 + d\rho$. For each region, the density is constant, and the gravitational potential for each infinitesimal region is Gm_i/R where m_i is the mass of the i^{th} region. Since the total mass M is Σm_i, the gravitational potential for the entire body is GM/R.

SECTION QUIZ

1. A region in space is bounded by $z = xy$, $z = x - y$, $y = 3$, $y = -x$, $x = 0$, and $x = 3$.
   (a) Compute the volume of the region.
   (b) Find the region's center of mass, assuming constant density.

2. A region is bounded by the planes $z = x + y$, $y = x + 1$, $x = 2$, $z = 0$, $y = 0$, and $x = 0$.

   (a) What is the average value of $x + y + z$ over this region?

   (b) If the density is xyz, what is the mass of the region?

3. To join the PTA (Party Troupers of America), one must demonstrate an aptitude for having a good time. Applicants are given three 100-point exams in the categories Eating, Drinking, and Merriment. The point scores (E , D , and M , respectively) are combined by the formula $S = \int_0^E\int_0^D\int_0^M (edm)^{-5/6} dm\, dd\, de$.

   (a) What is the combined perfect score, i.e., what is S when $E = D = M = 100$?

   (b) One needs 75% of the score in part (a) to qualify for admission into the PTA. Will 60 points in each category meet this requirement?

ANSWERS TO PREREQUISITE QUIZ

1. 1/2
2. (2/3,4/3)
3. 2/3
4. $\pi/4$

ANSWERS TO SECTION QUIZ

1. (a) 333/8

   (b) (327/185,147/37,57/37)

2. (a) 37/9

   (b) 5549/180

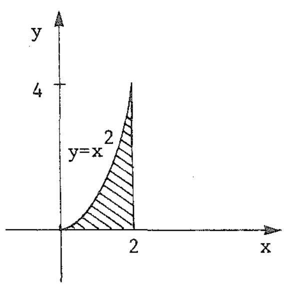
3. (a) 2160

   (b) Yes; it is over 77% .

17.R Review Exercises for Chapter 17

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $\int_2^3\int_4^8[x^3 + \sin(x + y)]\,dx\,dy = \int_2^3\{[x^4/4 - \cos(x + y)]|_{x=4}^{8}\}dy = \int_2^3[960 - \cos(8 + y) + \cos(4 + y)]\,dy = [960y - \sin(8 + y) + \sin(4 + y)]|_2^3 = 960 - \sin(11) + \sin(10) + \sin(7) - \sin(6) \approx 961.4$.

5. 

The region may be described as the following type 1 region: $0 \leqslant y \leqslant x^2$ and $0 \leqslant x \leqslant 2$. Thus, $\iint_D(x^3 + y^2x)\,dx\,dy = \int_0^2\int_0^{x^2}(x^3 + y^2x)\,dy\,dx = \int_0^2[(x^3y + y^3x/3)|_{y=0}^{x^2}]dx = \int_0^2(x^5 + x^7/3)dx = (x^6/6 + x^8/24)|_0^2 = 2^6/(2)(3) + 2^8/(2^3)(3) = 32/3 + 32/3 = 64/3$.

9. $\int_0^1\int_0^x\int_0^y xyz\,dz\,dy\,dx = \int_0^1\int_0^x (xyz^2/2)|_{z=0}^{y}\,dy\,dx = \int_0^1\int_0^x (xy^3/2)dy\,dx = \int_0^1(xy^4/8)|_{y=0}^{x}\,dz = \int_0^1(x^5/8)dx = (x^6/48)|_0^1 = 1/48$.

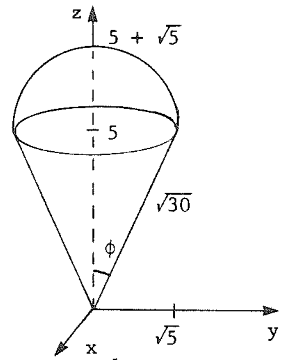
13. The volume of the solid is $\iiint_W dx\,dy\,dz$.

$$\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx =$$

$$2c\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\sqrt{1 - x^2/a^2 - y^2/b^2}\,dy\,dx .$$ Factor out $1/b$ to get

$$2c\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}(1/b)\sqrt{b^2 - b^2x^2/a^2 - y^2}\,dy\,dx .$$ Use formula 38 of the integration table to get $2c\int_{-a}^{a}\Big\{(1/b)[(y/2)\sqrt{b^2 - b^2x^2/a^2 - y^2} + ((b^2 - b^2x^2/a^2)/2)\sin^{-1}(y/\sqrt{b^2 - b^2x^2/a^2})]\Big|_{y=-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\Big\}dx =$

$2c\int_{-a}^{a}(1/b)((b^2 - b^2x^2/a^2)/2)\pi\,dx = \pi cb\int_{-a}^{a}(1 - x^2/a^2)dx = \pi bc(x - x^3/3a^2)|_{-a}^{a} = \pi bc(4a/3) = (4/3)\pi abc$.

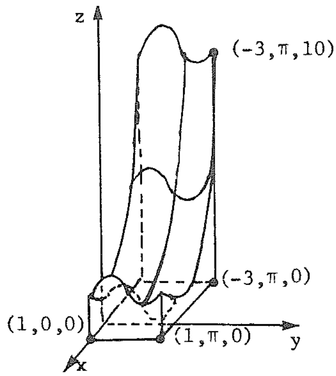
17. 

This volume may be found by adding the volume of the hemisphere of radius $\sqrt{5}$ to the volume of the cone formed by revolving a right triangle with sides 5 and $\sqrt{5}$. The volume of the cone is $\int_0^{2\pi}\int_0^{\cos^{-1}(5/\sqrt{30})}\int_0^{5/\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta =$ $\int_0^{2\pi}\int_0^{\cos^{-1}(5/\sqrt{30})}(\rho^3\sin\phi/3|_{\rho=0}^{5/\cos\phi})d\phi\,d\theta =$ $(1/3)\int_0^{2\pi}\int_0^{\cos^{-1}(5/\sqrt{30})}125\tan\phi\sec^2\phi\,d\phi\,d\theta$. Substitute $u = \sec\phi$ to get $(125/3)\int_0^{2\pi}[(\sec^2\phi/2)|_{\phi=0}^{\cos^{-1}(5/\sqrt{30})}]\,d\theta = (125/6)\int_0^{2\pi}(6/5 - 1)d\theta = 2\pi(25/6) =$ $25\pi/3$.

The calculations simplify if the sphere $x^2 + y^2 + (z - 5)^2 = 5$ is centered at the origin. Then the volume of the hemisphere is $\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{\sqrt{5}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_{\pi/2}^{\pi}[\rho^3\sin\phi/3|_{\rho=0}^{\sqrt{5}}]d\phi\,d\theta =$ $(5\sqrt{5}/3)\int_0^{2\pi}\int_{\pi/2}^{\pi}\sin\phi\,d\phi\,d\theta = (5\sqrt{5}/3)\int_0^{2\pi}[(-\cos\phi)|_{\phi=\pi/2}^{\pi}]d\theta =$ $(5\sqrt{5}/3)\int_0^{2\pi}d\theta = 10\sqrt{5}\pi/3$. Thus, the total volume is $(25 + 10\sqrt{5})\pi/3$.

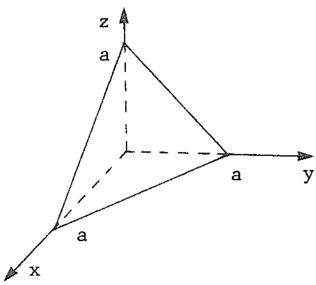
Alternatively, one may find it easier to use cylindrical coordinates to compute the volume of the cone or simply use $V = \pi r^2h/3$. Similarly, the volume of the sphere may be computed with $V = (1/2)(4\pi r^3/3)$.

21.



The volume under the graph of $f(x,y)$ is $\int_a^b\int_c^d f(x,y)\,dy\,dx$. $\int_{-3}^1\int_0^\pi [x^2 + \sin(2y) + 1]\,dy\,dx = \int_{-3}^1\{[y(x^2+1) - (1/2)\cos(2y)]|_{y=0}^{\pi}\}\,dx = \int_{-3}^1 \pi(x^2+1)dx = \pi(x^3/3 + x)|_{-3}^1 = 40\pi/3$.

25.



When the cut is made by the plane $x + y + z = a$, the volume of the solid below that plane is $\int_0^a\int_0^{a-x}\int_0^{a-x-y} dz\,dy\,dx = \int_0^a\int_0^{a-x}(a - x - y)dy\,dx = \int_0^a \{[(a-x)y - y^2/2]|_{y=0}^{a-x}\}\,dx = \int_0^a[(a-x)^2/2]\,dx$. Substitute $u = a - x$ to get $-(1/2)\int_{-a}^0 u^2du = -(1/2)(u^3/3)|_{-a}^0 = a^3/6$. Thus, the volume for the entire solid, when $a = 1$, is $1/6$. If the solid is to be cut into n equal volumes, then the volume under $x + y + z = a$ should be $k/6n$ where k is an integer such that $1 \leqslant k \leqslant n - 1$. Therefore, $a^3/6 = k/6n$ implies that cuts should be made in the planes $x + y + z = \sqrt[3]{k/n}$.

29. In cylindrical coordinates, the "dish" is described by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, $-\sqrt{25/4 - r^2} + 2 \leqslant z \leqslant 0$. The mass of the "dish" is $\rho\int_0^1\int_0^{2\pi}\int_{-\sqrt{25/4-r^2}+2}^{0} r\,dz\,d\theta\,dr = \rho\int_0^1\int_0^{2\pi}(r\sqrt{25/4 - r^2} - 2r)d\theta\,dr = 2\pi\rho\int_0^1(r\sqrt{25/4 - r^2} - 2r)dr$. Let $u = 25/4 - r^2$ to get $2\pi\rho\,[\int_{25/4}^{21/4}\sqrt{u}\,du/(-2) - \int_0^1 2r\,dr] = 2\pi\rho\,[-u^{3/2}/3|_{25/4}^{21/4} - r^2|_0^1] = \pi\rho(101 - 21\sqrt{21})/12$. The numerator of $\bar{x}$ is $\rho\int_0^1\int_0^{2\pi}\int_{-\sqrt{25/4-r^2}+2}^{0} r^2\cos\theta\,dz\,d\theta\,dr =$

29. (continued)

$\rho\int_0^1\int_0^{2\pi}(\sqrt{25/4 - r^2} + 2)r^2 \cos\theta \, d\theta \, dr = \rho\int_0^1[(\sqrt{25/4 - r^2} + 2) \times$ $r^2 \sin\theta|_{\theta=0}^{2\pi}] \, dr = 0$. The numerator of $\bar{y}$ is

$\rho\int_0^1\int_0^{2\pi}\int_{-\sqrt{25/4-r^2}+2}^{0} r^2\sin\theta \, dz \, d\theta \, dr = \rho\int_0^1\int_0^{2\pi}(\sqrt{25/4 - r^2} + 2) \times$ $r^2 \sin\theta \, d\theta \, dr = -\rho\int_0^1[(\sqrt{25/4 - r^2} + 2)r^2 \cos\theta|_{\theta=0}^{2\pi} \, dr = 0$. The numerator of $\bar{z}$ is $\rho\int_0^1\int_0^{2\pi}\int_{-\sqrt{25/4-r^2}+2}^{0} rz \, dz \, d\theta \, dr = \rho\int_0^1\int_0^{2\pi}(rz^2/$ $2\Big|_{z=-\sqrt{25/4-r^2}+2}^{0})d\theta \, dr = (\rho/2)\int_0^1\int_0^{2\pi} r(-41/4 + r^2 + 4\sqrt{25/4 - r^2})d\theta \, dr =$ $\pi\rho\int_0^1(-41r/4 + r^3 + 4r\sqrt{25/4 - r^2})dr$. Let $u = 25/4 - r^2$ to get $\pi\rho[(-41r^2/8 + r^4/4)|_0^1 - 2\int_{25/4}^{21/4}\sqrt{u} \, du] = \pi\rho[-39/8 - (4u^{3/2}/3)|_{25/4}^{21/4}] =$ $\pi\rho(383 - 84\sqrt{21})/24$. Therefore, the center of mass is $(0,0,(393 - 84\sqrt{21})/(202 - 42\sqrt{21})) \approx (0,0,-0.203)$.

33. The surface area is given by the formula $\iint_D\sqrt{1 + f_x^2 + f_y^2} \, dx \, dy$. In polar coordinates, D is described by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$. $f_x =$ $2x$ and $f_y = -2y$, so the integrand is $\sqrt{1 + 4x^2 + 4y^2} = \sqrt{1 + 4r^2}$. Therefore, the surface area is $\int_0^{2\pi}\int_0^1 r\sqrt{1 + 4r^2} \, dr \, d\theta$. Let $u =$ $1 + 4r^2$ to get $\int_0^{2\pi}\int_1^5 (\sqrt{u} \, du/8)d\theta = \int_0^{2\pi}(u^{3/2}/12|_{u=1}^5)d\theta = \pi(5\sqrt{5} - 1)/6$.

37. In polar coordinates, the region D is described by $1 \leqslant r \leqslant \sqrt{2}$ and $0 \leqslant \theta \leqslant \pi/2$, so the integral is $\int_0^{\pi/2}\int_1^{\sqrt{2}} (r/r^2)dr \, d\theta =$ $\int_0^{\pi/2}(\ln r|_{r=1}^{\sqrt{2}})d\theta = (\pi/2) \ln\sqrt{2} = (\pi/4) \ln(2)$.

41. In each case, the distance from (x,y) to ℓ , the x-axis, is y ; therefore, $D^2(x,y) = y^2$.

(a) $I = \int_{-1}^1\int_{-2}^2 y^2 \, dy \, dx = \int_{-1}^1 (y^3/3)|_{y=-2}^2 \, dx = \int_{-1}^1(16/3)dx =$ $(16x/3)|_{-1}^1 = 32/3$.

41. (b) In polar coordinates, we have $I = \int_0^4\int_0^{2\pi} (r \sin\theta)^2 r\, d\theta\, dr = \int_0^4\int_0^{2\pi} r^3 \sin^2\theta\, d\theta\, dr = \int_0^4\int_0^{2\pi} [r^3(1 - \cos 2\theta)/2]\, d\theta\, dr = \int_0^4 [r^3(\theta - \sin 2\theta/2)/2]\,\Big|_{\theta=0}^{2\pi} dr = \int_0^4 \pi r^3\, dr = (\pi r^4/4)\Big|_0^4 = 64\pi$.

45. (a) By definition, the average value of a general function is the volume divided by the area of the base. According to Exercise 26, the volume of a linear function over a rectangle is equal to the average of the heights of the four vertical edges times the area of the base. Therefore, dividing this volume by the area of the base implies that the average is just the average height of the four vertices.

(b) A parallelopiped and a box with the same lengths, widths, and heights have the same volumes. A parallelogram and a rectangle with equal lengths and widths have equal areas. Therefore, in analogy with the explanation in part (a), the average value of a linear function on a parallelogram is equal to the average of the heights of the four vertices.

49. From the theorem in Exercise 48 with $g(x,y) = f_y(x,y)$, we get $(d/dx)\int_c^d f_y(x,y)\, dy = \int_c^d f_{xy}(x,y)\, dy$, i.e., $(d/dx)[f(x,d) - f(x,c)] = \int_c^d f_{xy}(x,y)\, dy$. Thus, $[f(b,d) - f(b,c)] - [f(a,d) - f(a,c)] = \int_a^b\int_c^d f_{xy}(x,y)\, dy\, dx$. The same argument with x and y interchanged gives the same left-hand side and so $\int_a^b\int_c^d f_{xy}(x,y)\, dy\, dx = \int_c^d\int_a^b f_{yx}(x,y)\, dx\, dy$, i.e., the double integrals of f_{xy} and f_{yx} are the same. Since this can be done over any rectangle, f_{xy} and f_{yx} must be the same. (One invokes here the fact that if the double integral of a continuous function over arbitrarily small rectangles is 0 , then the function is 0 .)

TEST FOR CHAPTER 17

1. True or false.

   (a) $\int_0^\infty \exp(-x^2)dx = \sqrt{\pi}/2$.

   (b) If $f(x,y) \geqslant 0$ for all x and y , and $b > a > 0$, $d > c > 0$, then $\int_a^b\int_c^d f(x,y)dy\ dx = \int_a^b\int_c^d\int_0^{f(x,y)} dz\ dy\ dx$.

   (c) The mass of a solid W may be computed by using the triple integral $\iiint_W f(x,y,z)dx\ dy\ dz$, where $f(x,y,z)$ is the density of the solid at (x,y,z) .

   (d) As long as f is integrable over W , it is always true that $\iiint_W f(x,y,z)dx\ dy\ dz = \iiint_W f(x,y,z)dz\ dx\ dy$.

   (e) The geometric interpretation of the double integral $\iint_D dy\ dx$ is either (i) the area of D or (ii) the volume of the region between the planes $z = 0$ and $z = 1$, and bounded by the boundary of D .

2. Let A be the region bounded by the curve $y = x^4$ and the lines $y = |x^3|$

   (a) Write the area of A as a double integral, integrating in y first.

   (b) Rewrite the area of A by reversing the order of integration.

   (c) Compute the area of A .

3. (a) Compute $\int_{-\infty}^{\infty} 2\pi \exp(-9x^2)dx$.

   (b) Compute $\int_{-1}^1\int_3^5\int_0^{x^2} dy\ dx\ dz$ and interpret your answer geometrically.

4. Find the average value of $\ln z$ over the region bounded by $z = 1$, $z = xy$, $x = 1$, $x = 3y$, $y = 2$, and $y = 3$.

5. Find the volume between the cone $9 - x^2 - y^2$ and the xy-plane, where the domain is $x^2 + y^2 \leqslant 16$.

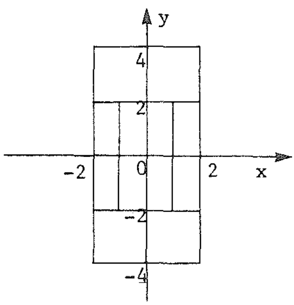
6. Find the center of mass of the region described by $x \leqslant y \leqslant 3x$, $0 \leqslant x \leqslant 2$, and $-y \leqslant z \leqslant 0$.

7. Compute the volume which lies inside both the sphere $x^2 + y^2 + z^2 = 25$ and the cylinder $y^2 + z^2 = 4$.

8. (a) Express the surface area of $f(x,y) = xy + 3$ over the rectangle $[1,3] \times [0,2]$ as a double integral. (Do not evaluate.)

(b) Express the surface area of $g(x,y) = (x^2 + y^2)/2 + 8$ over the ellipse $x^2 + 4y^2 = 1$ as a double integral. (Do not evaluate.)

(c) What is the relationship between the surface area of $f(x,y)$ and $g(x,y)$ over the same domain?

9. 

We wish to find the volume of the semi-ellipsoidal region described by $4x^2 + y^2 + 16z^2 = 16$ and $z \geqslant 0$. Use the grid shown at the left to make your estimates.

(a) Use the maximum of z on each rectangle of the domain to compute an upper sum.

(b) Use the minimum of z on each rectangle of the domain to compute a lower sum.

(c) How is the actual volume V related to your answers in (a) and (b)?

(d) Compute the actual volume to verify your answer in (c).

10. Recent research has shown that the number of additional strands of hair lost weekly is given by $\int_0^X\int_0^Y \sqrt{x}\, dy\, dx$, where X is the number of hours spent studying during the week and Y is the number of hours spent scratching parasites out of one's hair during the week. The theory is

10. (continued)

that studying causes the brain to bulge which causes hair roots to pop out, and hair is pulled accidentally by scratching.

(a) Suppose a young college student spends 70 hours studying during finals' week and 14 hours scratching bugs from unwashed hair. How many additional strands of hair can this student expect to lose?

(b) Compare the answer in (a) to a normal week when studying occupies 12 hours and scratching occupies 3 hours.

ANSWERS TO CHAPTER TEST

1. (a) True

   (b) True

   (c) True

   (d) True

   (e) True

2. (a) $2\int_0^1\int_{x^4}^{x^3} dy\, dx$

   (b) $2\int_0^1\int_{y^{1/3}}^{y^{1/4}} dx\, dy$

   (c) 1/10

3. (a) $2\pi\sqrt{\pi}/3$

   (b) 196/3 ; volume of region described by $0 \leqslant y \leqslant x^2$, $3 \leqslant x \leqslant 5$, and $-1 \leqslant z \leqslant 1$.

4. $(2007 \ln 3 - 272 \ln 2 - 1048)/523$

5. 57π

6. $(3/8, 13/4, -13/8)$

7. $(500 - 84\sqrt{21})(\pi/3)$

8. (a) $\int_1^3\int_0^2 \sqrt{1 + x^2 + y^2}\, dy\, dx$

(b) $\int_{-1}^{1}\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \sqrt{1 + x^2 + y^2}\, dy\, dx$

(c) Equal

9. (a) $12\sqrt{3} + 8$

(b) $2\sqrt{2}$

(c) $2\sqrt{2} \leqslant V \leqslant 12\sqrt{3} + 8$

(d) 8π

10. (a) $1960\sqrt{70}/3$

(b) $24\sqrt{12}$